6.1 Basics of counting

- **Combinatorics**: they study of arrangements of objects
- **Enumeration**: the counting of objects with certain properties (an important part of combinatorics)
  – Enumerate the different telephone numbers possible in US
  – The allowable password on a computer
  – The different orders in which runners in a race can reach

Example

- Suppose a password on a system consists of 6, 7, or 8 characters
- Each of these characters must be a digit or a letter of the alphabet
- Each password must contain at least one digit
- How many passwords are there?

Basic counting principles

- Two basic counting principles
  - Product rule
  - Sum rule
- **Product rule**: suppose that a procedure can be broken down into a sequence of two tasks
- If there are n_1 ways to do the 1st task, and each of these there are n_2 ways to do the 2nd task, then there are $n_1 \cdot n_2$ ways to do the procedure

Example

- The chairs of a room to be labeled with a letter and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?
- There are 26 letters to assign for the 1st part and 100 possible integers to assign for the 2nd part, so there are $26 \cdot 100 = 2600$ different ways to label chairs

Product rule

- Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \ldots, T_m$ in sequence. If each task T_i, $i=1, 2, \ldots, n$ can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_m$ ways to carry out the procedure

Example

- How many different license plates are available if each plate contains a sequence of 3 letters followed by 3 digits (and non sequences of letters are prohibited, even if they are obscene)?
- License plate _ _ _   _ _ _ : There are 26 choices for each letter and 10 choices for each digit. So, there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ possible license plates

Counting functions

- How many functions are there from a set with m elements to a set with n elements?
- A function corresponds to one of the n elements in the codomain for each of the m elements in the domain
- Hence, by product rule there are $n \cdot n \ldots \cdot n = n^m$ functions from a set with m elements to one with n elements

Counting one-to-one functions

- How many one-to-one functions are there from a set with m elements to one with n elements?
- First note that when $m>n$ there are no one-to-one functions from a set with m elements to one with n elements
- Let $m \leq n$. Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value for the value at a_1
- As the function is one-to-one, the value of the function at a_2 can be picked in n-1 ways (the value used for a_1 cannot be used again)
- Using the product rule, there are $n(n-1)(n-2)\ldots(n-m+1)$ one-to-one functions from a set with m elements to one with n elements

Example

- From a set with 3 elements to one with 5 elements, there are $5 \cdot 4 \cdot 3 = 60$ one-to-one functions

Example

- The format of telephone numbers in north America is specified by a numbering plan
- It consists of 10 digits, with 3-digit area code, 3-digit office code and 4-digit station code
- Each digit can take one form of
  - X: 0, 1, …, 9
  - N: 2, 3, …, 9
  - Y: 0, 1

Example

- In the old plan, the formats for area code, office code, and station code are NYX, NNX, and XXXX, respectively
- So the phone numbers had NYX-NNX-XXXX
- NYX: $8 \cdot 2 \cdot 10 = 160$ area codes
- NNX: $8 \cdot 8 \cdot 10 = 640$ office codes
- XXXX: $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes
- So, there are $160 \cdot 640 \cdot 10{,}000 = 1{,}024{,}000{,}000$ phone numbers

X: 0, 1, …, 9
N: 2, 3, …, 9
Y: 0, 1

Example

- In the new plan, the formats for area code, office code, and station code are NXX, NXX, and XXXX, respectively
- So the phone numbers had NXX-NXX-XXXX
- NXX: $8 \cdot 10 \cdot 10 = 800$ area codes
- NXX: $8 \cdot 10 \cdot 10 = 800$ office codes
- XXXX: $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes
- So, there are $800 \cdot 800 \cdot 10{,}000 = 6{,}400{,}000{,}000$ phone numbers

Product rule

- If A_1, A_2, …, A_m are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set
- $|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \times |A_2| \times \ldots \times |A_m|$

Sum rule

- If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are n_1+n_2 ways to do the task
- Example: suppose either a member of faculty or a student in CSE is chosen as a representative to a university committee. How many different choices are there for this representative if there are 8 members in faculty and 200 students?
- There are 8+200=208 ways to pick this representative

Sum rule

- If A_1, A_2, ..., A_m are disjoint finite sets, then the number of elements in the union of these sets is as follows

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|$$

More complex counting problems

- In a version of the BASIC programming language, the name of a variable is a string of 1 or 2 alphanumeric characters, where uppercase and lowercase letters are not distinguished.
- Moreover, a variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use
- How many different variables names are there?
- Let V_1 be the number of these variables of 1 character, and likewise V_2 for variables of 2 characters
- So, $V_1=26$, and $V_2=26 \cdot 36-5=931$
- In total, there are $26+931=957$ different variables

Example

- Each user on a computer system has a password, which is 6 to 8 characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?
- Let P be the number of all possible passwords and $P=P_6+P_7+P_8$ where P_i is a password of i characters
- $P_6=36^6-26^6=1,867,866,560$
- $P_7=36^7-26^7=70,332,353,920$
- $P_8=36^8-26^8=208,827,064,576$
- $P=P_6+P_7+P_8=2,684,483,063,360$

Example: Internet address

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Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid				hostid			
Class B	1	0	netid				hostid		
Class C	1	1	0	netid				hostid	
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

- Internet protocol (IPv4)
  - Class A: largest network
  - Class B: medium-sized networks
  - Class C : smallest networks
  - Class D: multicast (not assigned for IP address)
  - Class E: future use
  - Some are reserved: netid 1111111, hostid all 1’s and 0’s
- Neither class D or E addresses are assigned as the IPv4 addresses
- How may different IPv4 addresses are available?

Example: Internet address

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Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid				hostid			
Class B	1	0	netid				hostid		
Class C	1	1	0	netid				hostid	
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

- Let the total number of address be x, and $x=x_A+x_B+x_C$
- Class A: there are 2^7-1=127 netids (1111111 is reserved). For each netid, there are 2^{24}-2=16,777,214 hostids (as hostids of all 0s and 1s are reserved), so there are x_A=127·16,777,214=2,130,706,178 addresses
- Class B, C: 2^{14}=16,384 Class B netids and 2^{21}=2,097,152 Class C netids. 2^{16}-2=65,534 Class B hostids, and 2^8-2=254 Class C hostids. So, x_B=1,073,709,056, and x_C=532,676,608
- So, $x=x_A+x_B+x_C$=3,737,091,842

Inclusion-exclusion principle

- Suppose that a task can be done in n_1 or in n_2 ways, but some of the set of n_1 ways to do the task are the same as some of the n_2 ways to do the task
- Cannot simply add n_1 and n_2, but need to subtract the number of ways to the task that is common in both sets
- This technique is called **principle of inclusion-exclusion** or **subtraction principle**

Example

- How many bit strings of length 8 either start with a 1 or end with two bits 00?
- 1 _ _ _ _ _ _ _: 2^7=128 ways
- _ _ _ _ _ _ 00: 2^6=64 ways
- 1 _ _ _ _ _ 00: 2^5=32 ways
- Total number of possible bit strings is 128+64-32=160

Inclusion-exclusion principle

- Using sets to explain

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Tree diagrams

- How many bit strings of length 4 do not have two consecutive 1s?
- In some cases, we can use tree diagrams for counting

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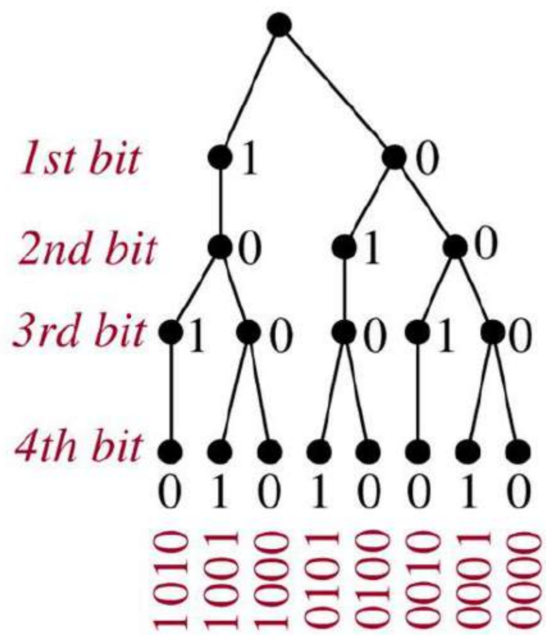

8 without two consecutive 1s

Example

- A playoff between 2 teams consists of at most 5 games. The 1st team that wins 3 games wins the playoff. How many different ways are there?

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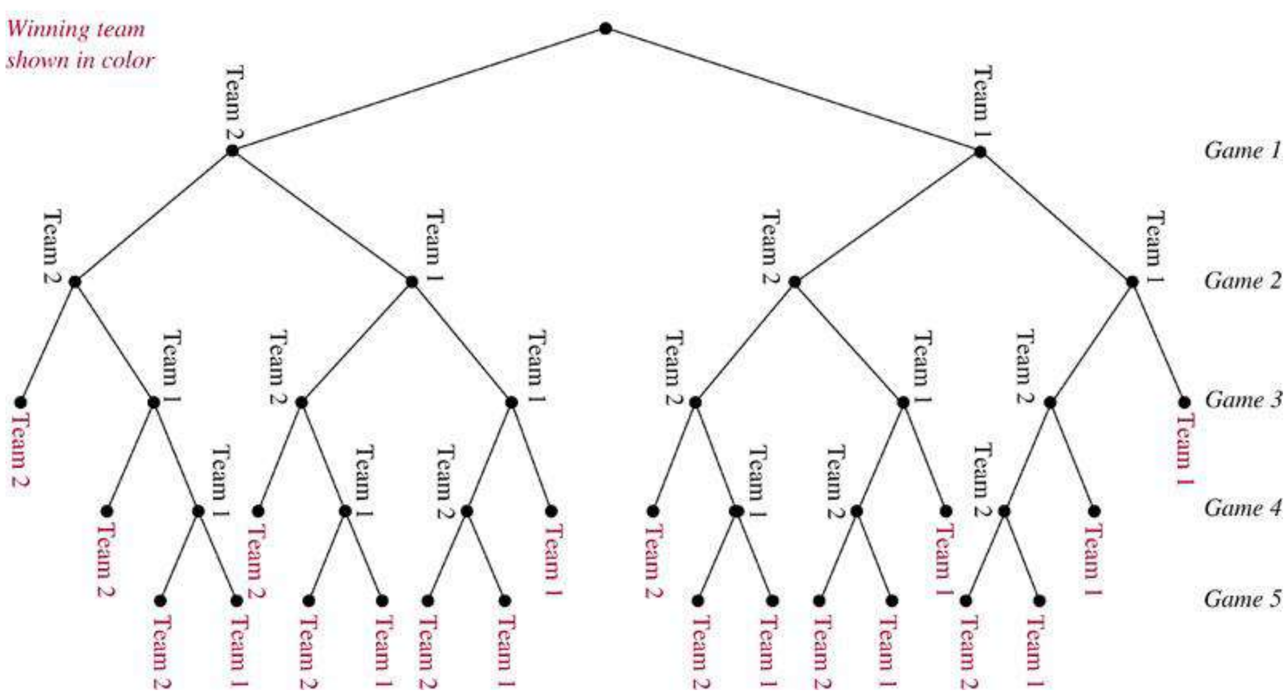

Example

- Suppose a T-shirt comes in 5 different sizes: S, M, L, XL, and XXL. Further suppose that each size comes in 4 colors, white, green, red, and black except for XL which comes only in red, green and black, and XXL which comes only in green and black. How many possible size and color of the T-shirt?

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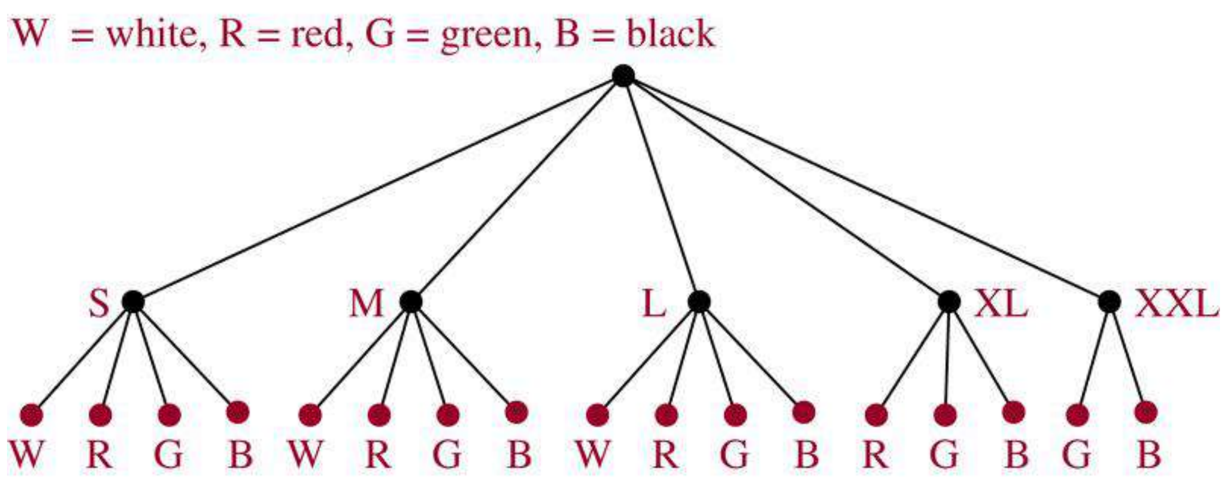

6.2 Pigeonhole principle

- Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost
- Thus, **at least** 1 of these 19 pigeonholes must have **at least** 2 pigeons
- Why? If each pigeonhole had at most one pigeon in it, at most 19 pigeons, 1 per hole, could be accommodated
- If there are more pigeons than pigeonholes, then there must be at least 1 pigeonhole with at least 2 pigeons in it

Example

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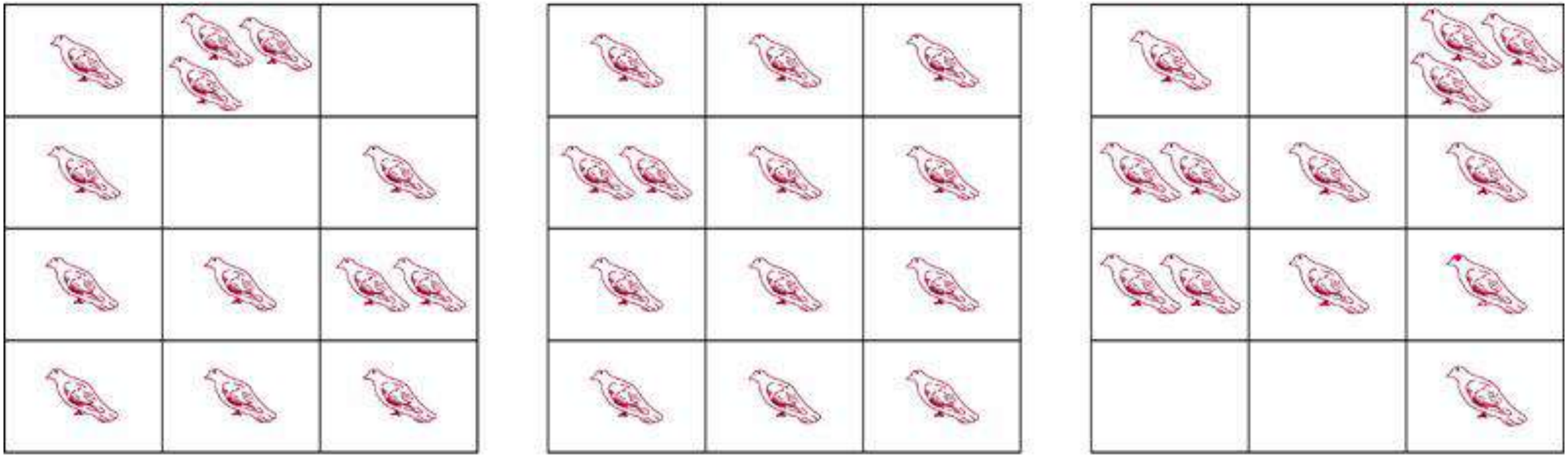

(a) (b) (c)

13 pigeons and 12 pigeonholes

Pigeonhole principle

- Theorem 1: If k is a positive integer and k+1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects
- Proof: suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction as there are at least k+1 objects
- Also known as **Dirichlet drawer principle**

Pigeonhole principle

- Corollary 1: A function f from a set with k+1 or more elements to a set with k elements is not one-to-one
- Proof: Suppose that for each element y in the codomain of f we have a box that contains all elements x of the domain f s.t. f(x)=y
- As the domain contains k+1 or more elements and the codomain contain only k elements, the pigeonhole principle tells us that one of these boxes contains 2 or more elements x of the domain
- This means that f cannot be one-to-one

Example

- Among any group of 367 people, there must be at least 2 with the same birthday
- How many students must be in a class to guarantee that at least 2 students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points

Generalized pigeonhole principle

- Theorem 2: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects
- Proof: Proof by contradiction. Suppose that none of the boxes contains more than $\lceil N/k \rceil -1$ objects. Then the total number of objects is at most $k(\lceil N/k \rceil -1) < k((N/k+1)-1) = N$

  where the inequality $\lceil N/k \rceil < N/k+1$ is used
- This is a contradiction as there are a total of N objects

Generalized pigeonhole principle

- A common type of problem asks for the minimum number of objects s.t. at least r of these objects must be in one of k boxes when these objects are distributed among boxes
- When we have N objects, the generalized pigeonhole principle tells us there must be at least r objects in one of the boxes as long as $\lceil N/k \rceil \geq r$. Recall $N/k+1 > \lceil N/k \rceil$. The smallest integer N with $N/k > r-1$, i.e., $\mathbf{N=k(r\text{-}1)+1}$ is the smallest integer satisfying the inequality $\lceil N/k \rceil \geq r$

Example

- Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month
- What is the minimum number of students required in a discrete mathematics class to be sure that at least 6 will receive the same grade, if there are 5 possible grades,?
- The minimum number of studeA, B, C, D, and Fnts needed to ensure at least 6 students receive the same grade is the smallest integer N s.t. $\lceil N/5 \rceil = 6$. Thus, the smallest $N = 5 \cdot 5 + 1 = 26$

Example

- How many cards must be selected from a standard deck of 52 cards to guarantee that a least 3 cards of the same suit are chosen?
- Suppose there are 4 boxes, one for each suit. If N cards are selected, using the generalized pigeonhole principle, there is at lest one box containing at least $\lceil N/4 \rceil$ cards
- Thus to have $\lceil N/4 \rceil \geq 3$, the smallest N is $2 \cdot 4 + 1 = 9$. So at least 9 cards need to be selected

Example

- How many cards must be selected to guarantee that at least 3 hearts are selected?
- We do not use the generalized pigeonhole principle to answer this as we want to make sure that there are 3 hearts, not just 3 cards of one suit
- Note in the worst case, we can select all the clubs, diamonds, and spades, 39 cards in all before selecting a single heart
- The next 3 cards will be all hearts, so we may need to select 42 cars to guarantee 3 hearts are selected

Applications of Pigeonhole principle

- During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games
- Let a_j be the number of games played on or before jth day of the month. Then a_1, a_2, …, a_{30} is an increasing sequence of distinctive positive integers with $1 \leq a_j \leq 45$. Moreover a_1+14, a_2+14, …, $a_{30}+14$ is also an increasing sequence of distinct positive integers with $15 \leq a_j+14 \leq 59$
- The 60 positive integers, a_1, a_2, …, a_{30}, a_1+14, a_2+14, …, $a_{30}+14$ are all less than or equal to 59. Hence, by the pigeonhole principle, two of these integers must be equal, i.e., there must be some I and j with $a_i=a_j+14$. This means exactly 14 games were played from day j+1 to day i

Ramsey theory

- Example: Assume that in a group of 6 people, each pair of individuals consists of two friends or 2 enemies. Show that there are either 3 mutual friends or 3 mutual enemies in the group
- Let A be one of the 6 people. Of the 5 other people in the group, there are either 3 or more who are friends of A, or 3 or more are enemies of A
- This follows from the generalized pigeonholes principles, as 5 objects are divided into two sets, one of the sets has at least $\lceil 5/2 \rceil = 3$ elements

Ramsey number

- **Ramsey number** R(m, n) where m and n are positive integers greater than or equal to 2, denotes the minimum number of people at a party s.t. there are either m mutual friends or n mutual enemies, assuming that every pair of people at the party are friends or enemies
- In the previous example, $R(3,3) \leq 6$
- We conclude that $R(3,3)=6$ as in a group of 5 people where every two people are friends or enemies, there may not be 3 mutual friends or 3 mutual enemies

6.3 Permutations & Combinations

- Counting:
  - Find out the number of ways to select a particular number of elements from a set
  - Sometimes the order of these elements matter
- Example:
  - How many ways we can select 3 students from a group of 5 students?
  - How many different ways they stand in line for picture?

Permutation

- How many ways can we select 3 students from a group of 5 to stand in lines for a picture?
- First note that the order in which we select students matters
- There are 5 ways to select the 1st student
- Once the 1st student is selected, there are 4 ways to select the 2nd student in line. By product rule, there are 5x4x3=60 ways to select 3 student from a group of 5 students to stand line for picture

Permutation

- How many ways can we arrange all 5 in a line for a picture?
- By product rule, we have 5x4x3x2x1=120 ways to arrange all 5 students in a line for a picture

Permutation

- A **permutation** of a set of distinct objects is an **ordered** arrangement of these objects
- An ordered arrangement of r elements of a set is called an r-permutation
- The number of r-permutation of a set with n element is denoted by P(n,r). We can find P(n,r) using the product rule
- Example: Let S={1, 2, 3}. The ordered arrangement 3, 1, 2 is a permutation of S. The ordered arrangement 3, 2, is a 2-permutation of S

Permutation

- Let S={a, b, c}. The 2-permutation of S are the ordered arrangements, a, b; a, c; b, a; b, c; c, a; and c, b
- Consequently, there are 6 2-permutation of this set with 3 elements
- Note that there are 3 ways to choose the 1^{st} element and then 2 ways to choose the 2^{nd} element
- By product rule, there are $P(3,2)=3 \times 2 = 6$

r-permutation

- Theorem 1: If n is a positive and r is an integer with $1 \leq r \leq n$, then there are

  $P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$

  r-permutations of a set with n elements
- Proof: Use the product rule, the first element can be chosen in n ways. There are n-1 ways to chose the 2^{nd} element. Likewise, there are n-2 ways to choose 3^{rd} element, and so on until there are exactly $n-(r-1)=n-r+1$ ways to choose the r-th element. Thus, there are $n \cdot (n-1) \cdot (n-2) \ldots \cdot (n-r+1)$ r-permutations of the set

r-permutation

- Note that $p(n,0)=1$ whenever n is a nonnegative integer as there is exactly one way to order zero element
- Corollary 1: If n and r are integers with $0 \leq r \leq n$, then $P(n,r)=n!/(n-r)!$
- Proof: When n and r are integers with $1 \leq r \leq n$, by Theorem 1 we have

$$P(n,r)=n(n-1)\ldots(n-r+1)=n!/(n-r)!$$

- As $n!/(n-0)!=1$ when n is a nonnegative integer, we have $P(n,r)=n!/(n-r)!$ also holds when $r=0$

r-permutation

- By Theorem 1, we know that if n is a positive integer, then $P(n,n)=n!$
- Example: How many ways are there to select a 1st prize winner, a 2nd prize winner, and a 3rd prize winner from 100 different contestants?
- $P(100,3)=100 \times 99 \times 98=970,200$

Example

- How many permutations of the letters ABCDEFGH contain string ABC?
- As ABC must occur as a block, we can find the answer by finding the permutations of 6 letters, the block ABC and the individual letters, D,E,F,G, and H. As these 6 objects must occur in any order, there are 6!=720 permutations of the letters ABCDEFGH in which ABC occurs in a block

Combinations

- How many different committees of 3 students can be formed from a group of 4 students?
- We need to find the number of subsets with 3 elements from the set containing 4 students
- We see that there are 4 such subsets, one for each of the 4 students as choosing 4 students is the same as choosing one of the 4 students to leave out of the group
- This means there are 4 ways to choose 3 students for the committee, where th order in which these students are chosen does not matter

r-combination

- An r-combination of elements of a set is an **unordered** selection of r elements from the set
- An r-combination is simply a subset of the set with r elements
- Denote by C(n,r). Note that C(n,r) is also denoted by $\binom{n}{r}$ and is called a binomial coefficient

Example

- Let S be the set {1, 2, 3, 4}. Then {1, 3, 4} is a 3-combination from S
- We see that C(4,2)=6, as the 2-combination of {a, b, c, d} are 6 subsets {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, and {c, d}

r-combination

- We can determine the number of r-combinations of a set with n elements using the formula for the number of r-permutations of a set
- Note that the r-permutations of a set can be obtained by first forming r-combinations and then ordering the elements in these combinations

r-combination

- The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$ equals

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

- Proof: The r-permutations of the set can be obtained by forming the C(n,r) r-combinations and then ordering the elements in each r-permutation which can be done in P(r,r) ways

$$P(n,r) = C(n,r) \cdot P(r,r)$$

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

r-combination

- When computing r-combination

$$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{P(n,r)}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r!}$$

thus canceling out all the terms in the larger factorial

Example

- How many poker hands of 5 cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?
- Choose 5 out of 52 cards: C(52,5)=52!/(5!47!)= (52x51x50x49x48)/(5x4x3x2x1)=26x17x10x49 x12=2,598,960
- C(52,47)=52!/(47!5!)=2,5,98,960

Corollary 2

- Let n and r be nonnegative integers with r≤n. Then C(n,r)=C(n,n-r)
- Proof:

$$C(n,r)=\frac{n!}{r!(n-r)!}$$

$$C(n,n-r)=\frac{n!}{(n-r)!(n-(n-r))!}=\frac{n!}{(n-r)!r!}$$

Combinatorial proof

- A **combinatorial proof** of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways
- Proof of Corollary 2: Suppose that S is a set with n elements. Every subset A of S with r elements corresponds to a subset of S with n-r elements, i.e., $\overline{A}$. Thus, C(n,r)=C(n,n-r)

Example

- How many ways are there to select 5 players from a 10-member tennis team?
- Choose 5 out of 10 elements, i.e., $C(10, 5)=10!/(5!5!)=252$
- How many bit strings of length n contain exactly r 1s?
- This is equivalent to choose r elements from n elements, i.e., $C(n,r)$

6.4 Binomial coefficients

- The number of r-combinations form a set with n elements is often denoted by $\binom{n}{r}$
- Also called as a binomial coefficient as these numbers occur as coefficients in the expansion of powers of binomial expressions such as $(a+b)^n$
- A binomial expression is simply the sum of two terms, such as x+y

Example

- The expansion of $(x+y)^3$ can be found using combinational reasoning instead of multiplying the there terms out
- When $(x+y)^3=(x+y)(x+y)(x+y)$ is expanded, all products of a term in the 1st sum, a term in the 2nd sum, and a term in the 3rd sum are added, e.g., x^3, x^2y, xy^2, and y^3
- To obtain a term of the form x^3, an x must be chosen in each of the sums, and this can be done in only one way. Thus, the x^3 term in the product has a coefficient of 1

Example

- To obtain a term of the form x^2y, an x must be chosen in 2 of the 3 sums (and consequently a y in the other sum). Hence, the number of such terms is the number of 2-combinations of 3 objects, namely, $\binom{3}{2}$
- Similarly, the number of terms of the form xy^2 is the number of ways to pick 1 of the 3 sums to obtain an x (and consequently take a y from each of the other two sums), which can be done in $\binom{3}{1}$ ways

Example

- Consequently,

$$(x+y)^3 = (x+y)(x+y)(x+y) = (xx+xy+yx+yy)(x+y)$$
$$= xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy$$
$$= x^3+3x^2y+3xy^2+y^3$$

- The binomial theorem: Let x and y be variables, and let n be a nonnegative integer

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$
$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Binomial theorem

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

- Proof: A combinatorial proof of the theorem is given. The terms in the product when it is expanded are of the form $x^{n-j}y^j$ for j=0,1,2…, n
- To count the number of terms of the form $x^{n-j}y^j$, note that to obtain such a term it is necessary to choose n-j x's from the n sums (so that the other j terms in the product are y's). Therefore, the coefficients of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which is equal to $\binom{n}{j}$

Example

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$$

$$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$

- What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

$$\binom{25}{13} = \frac{25!}{13!12!} = 5{,}200{,}300$$

Example

- What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

$$(2x-3y)^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^{j}$$

- Consequently, the coefficient of $x^{12}y^{13}$ is

$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25!}{13!12!} 2^{12} 3^{13}$$

Corollary

- Corollary 1: Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

- Proof: Using Binomial theorem with x=1 and y=1

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$$

Corollary

- There is also a combinatorial proof of Corollary 1
- Proof: A set with n elements has a total of 2^n different subsets. Each subset has 0 elements, 1 element, 2 elements, or n elements in it. Thus, there are $\binom{n}{0}$ subsets with 0 elements, $\binom{n}{1}$ subsets with 1 element, … and $\binom{n}{n}$ subsets with n elements. Thus $\sum_{k=0}^{n}\binom{n}{k}$ counts the total number of subsets of a set with n elements,
- This shows that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$

Corollary 2

Let n be a positive integer. Then

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$$

$$\text{Proof}: 0=0^n=((-1)+1)^n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k 1^{n-k}=\sum_{k=0}^{n}\binom{n}{k}(-1)^k$$

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots$$

Corollary 3

Let n be a non-negative interger. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

$$\text{Proof}: (2+1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

Pascal identity and triangle

- Pascal's identity: Let *n* and *k* be positive integers with $n \geq k$. Then $$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$
- Combinatorial proof: Let T be a set containing *n+1* elements. Let a be an element in T, and let S=T-{*a*} (S has n elements)
- Note that there are $\binom{n+1}{k}$ subsets of T containing *k* elements. However, a subset of T with k elements either contains *a* (i.e., a subset of *k-1* elements of S) or not.
- There are $\binom{n}{k-1}$ subsets of *k-1* elements of S, and so there are $\binom{n}{k-1}$ subsets of *k-1* elements containing *a*
- There are $\binom{n}{k}$ subsets of k elements of T that do not contain *a*
- Thus $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Pascal identity and triangle

- Can also prove the Pascal identity with algebraic manipulation of $\binom{n}{k}$

$$\binom{n+1}{k} = \frac{(n+1)!}{k!(n-k+1)!}$$

$$\binom{n}{k-1} = \frac{n!}{(k-1)!(n-k+1)!} = \frac{kn!}{(k-1)!k(n-k+1)!} = \frac{kn!}{k!(n-k+1)!}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n-k+1)n!}{k!(n-k)!(n-k+1)} = \frac{(n-k+1)n!}{k!(n-k+1)!}$$

$$\binom{n}{k-1} + \binom{n}{k} = \frac{kn!}{k!(n-k+1)!} + \frac{(n-k+1)n!}{k!(n-k+1)!} = \frac{(k+n-k+1)n!}{k!(n-k+1)!} = \frac{(n+1)n!}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n-k+1)!}$$

Pascal’s triangle

- Pascal’s identity is the basis for a geometric arrangement of the binomial coefficients in a triangle

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$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \\
\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7} \\
\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8}
\end{array}
$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$
\begin{array}{c}
1 \\
1\quad 1 \\
1\quad 2\quad 1 \\
1\quad 3\quad 3\quad 1 \\
1\quad 4\quad 6\quad 4\quad 1 \\
1\quad 5\quad 10\quad 10\quad 5\quad 1 \\
1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1 \\
1\quad 7\quad 21\quad 35\quad 35\quad 21\quad 7\quad 1 \\
1\quad 8\quad 28\quad 56\quad 70\quad 56\quad 28\quad 8\quad 1
\end{array}
$$

(b)

each n-th row, binomial coefficients

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$